

# Gel'fand-Dorfman Bialgebras

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## 1 Introduction

A Gel'fand-Dorfman bialgebra is a vector space with a Lie algebra structure and a Novikov algebra structure, satisfying a certain compatibility condition. This bialgebraic structure corresponds to a certain Hamiltonian pairs in the Gel'fand-Dikii-Dorfman's theory of Hamiltonian operators (cf. [GDi], [GDo]). In this talk, I will give a survey on the study of Gel'fand-Dorfman bialgebras. First let me give a more technical initial introduction to the bialgebras through what I call "Lie algebra with one-variable structure."

Let  $\mathbb{F}$  be an arbitrary field. Denote by  $\mathbb{Z}$  the ring of integers and by  $\mathbb{N}$  the additive semi-group of nonnegative integers. All the vector spaces are assumed over  $\mathbb{F}$ . Let  $V$  be a vector space and let  $t$  be an indeterminant. Form a tensor

$$\hat{V} = V \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]. \quad (1.1)$$

We denote

$$u(z) = \sum_{n \in \mathbb{Z}} (u \otimes t^n) z^{-n-1} \quad \text{for } u \in V, \quad (1.2)$$

where  $z$  is a formal variable. A *Lie algebra with one-variable structure* is a vector space  $\hat{V}$  with the Lie bracket of the form

$$[u(z_1), v(z_2)] = \sum_{i=0}^m \sum_{j=0}^n z_2^{-1} \partial_{z_1}^i \delta\left(\frac{z_1}{z_2}\right) \partial_{z_2}^j w_{ij}(z_2) \quad (1.3)$$

for  $u, v \in V$ , where  $m, n$  are nonnegative integers depending on  $u, v$ , and  $w_{ij} \in V$ . Here we have used the notation

$$\delta(z) = \sum_{k \in \mathbb{Z}} z^k. \quad (1.4)$$

Moreover, each  $w_{ij}$  depends on  $u$  and  $v$  bilinearly by (1.2) and (1.3). Above definition is not in general form. We aim at giving the audience a simple rough picture. According to Wightman's axioms of quantum field theory, the algebraic content of two-dimensional quantum field theory is a certain representation theory of Lie algebras with one-variable structure (e.g., cf. [K]), where intertwining operators among irreducible modules, "partition

functions” (characters in algebraic terms) and “correlation functions” related to (1.3) etc. play important roles.

Suppose that  $m = n = 0$  in (1.3). We denote

$$w_{00} = [u, v]. \quad (1.5)$$

Then  $(V, [u, v])$  forms a Lie algebra and  $\hat{V}$  is the corresponding *loop algebra* with the Lie bracket

$$[u \otimes t^j, v \otimes t^k] = [u, v] \otimes t^{j+k} \quad \text{for } u, v \in V, j, k \in \mathbb{Z}. \quad (1.6)$$

If  $V = \mathbb{F}e$  is one-dimensional and

$$[e(z_1), e(z_2)] = z_2^{-1} \delta \left( \frac{z_1}{z_2} \right) \partial_{z_2} e(z_2) - 2z_2^{-1} \partial_{z_1} \delta \left( \frac{z_1}{z_2} \right) e(z_2), \quad (1.7)$$

Then  $\hat{V}$  is the *centerless Virasoro algebra* (or a rank-one Witt algebra) with the Lie bracket

$$[e \otimes t^j, e \otimes t^k] = (j - k) e \otimes t^{j+k} \quad \text{for } j, k \in \mathbb{Z}. \quad (1.8)$$

A natural generalization of the above one-dimensional case is

$$[u(z_1), v(z_2)] = z_2^{-1} \delta \left( \frac{z_1}{z_2} \right) \partial_{z_2} w_{01}(z_2) + z_2^{-1} \partial_{z_1} \delta \left( \frac{z_1}{z_2} \right) w_{10}(z_2) \quad (1.9)$$

for  $u, v \in V$ . Denote

$$w_{01} = v \circ u. \quad (1.10)$$

Then  $(V, \circ)$  is an algebra satisfying

$$(u \circ v) \circ w = (u \circ w) \circ v, \quad (1.11)$$

$$(u \circ v) \circ w - u \circ (v \circ w) = (v \circ u) \circ w - v \circ (u \circ w) \quad (1.12)$$

for  $u, v, w \in V$ . The above algebra  $(V, \circ)$  appeared in Gel’fand and Dorfman’s work [GDo], corresponding to certain Hamiltonian operators. Moreover, it also appeared in Balinskii and Novikov’s work [BN] as the local structures of certain Poisson brackets of hydrodynamic type. This structure was first abstractly studied by Zel’manov [Z], Filippov [F] and was named as “Novikov algebra” by Osborn [O1].

Note that (1.12) is the axiom of left-symmetric algebra. Left-symmetric algebras play fundamental roles in the theory of affine manifolds (cf. [A], [FD]). Novikov algebras are Left-symmetric algebras whose right multiplication operators are mutually commutative (cf. (1.11)). Furthermore,

$$w_{10} = -(u \circ v + v \circ u). \quad (1.13)$$

The Lie bracket on  $\hat{V}$  is

$$[u \otimes t^j, v \otimes t^k] = ju \circ v \otimes t^{j+k-1} - kv \circ u \otimes t^{j+k-1} \quad \text{for } u, v \in V, j, k \in \mathbb{Z}. \quad (1.14)$$

Consider the mixed case

$$[u(z_1), v(z_2)] = z_2^{-1} \delta \left( \frac{z_1}{z_2} \right) [w_{00}(z_2) + \partial_{z_2} w_{01}(z_2)] + z_2^{-1} \partial_{z_1} \delta \left( \frac{z_1}{z_2} \right) w_{10}(z_2) \quad (1.15)$$

for  $u, v \in V$ . Denote

$$w_{00} = [v, u], \quad w_{01} = v \circ u. \quad (1.16)$$

Then  $(V, [\cdot, \cdot])$  forms a Lie algebra,  $(V, \circ)$  forms a Novikov algebra and the following compatibility condition holds

$$[w \circ u, v] - [w \circ v, u] + [w, u] \circ v - [w, v] \circ u - w \circ [u, v] = 0 \quad (1.17)$$

for  $u, v, w \in V$ .

I called the triple  $(V, [\cdot, \cdot], \circ)$  a *Gel'fand-Dorfman bialgebra* (cf. [X4]), which corresponds to a certain Hamiltonian pair in Gel'fand and Dorfman's work [GDo]. Moreover, (1.13) holds. The Lie bracket on  $\hat{V}$  is

$$[u \otimes t^j, v \otimes t^k] = [v, u] \otimes t^{j+k} + ju \circ v \otimes t^{j+k-1} - kv \circ u \otimes t^{j+k-1} \quad (1.18)$$

for  $u, v \in V, j, k \in \mathbb{Z}$ .

For convenience, we shall use the following notation of index

$$\overline{m, n} = \{m, m+1, \dots, n\} \quad (1.19)$$

for  $m, n \in \mathbb{N}$  such that  $m \leq n$ .

The article is organized as follows. In Section 2, I will talk about structures of simple Novikov algebras and their irreducible representations. In Section 3, I will present some general constructions of Gel'fand-Dorfman bialgebras. In Section 4, I will give a few classification results on the bialgebras. In Section 5, examples of application to simple “cubic conformal algebras” and “quartic conformal algebras” will be given.

## 2 Novikov Algebras

Let us first give a construction of Novikov algebras. Let  $(\mathcal{A}, \cdot)$  be a commutative associative algebra. Then  $(\mathcal{A}, \cdot)$  forms a Novikov algebra, which is not so interesting. Take a derivation  $\partial$  of  $(\mathcal{A}, \cdot)$  and  $\xi \in \mathcal{A}$ . We define

$$u \circ_\xi v = u\partial(v) + \xi uv \quad \text{for } u, v \in \mathcal{A}. \quad (2.1)$$

Then  $(\mathcal{A}, \circ_\xi)$  forms a Novikov algebra. The above construction was found by S. Gel'fand when  $\xi = 0$  (cf. [GDo]), by Fillip [F] when  $\xi \in \mathbb{F}$  and by me [X2] in general case.

**Theorem 2.1 (Zel'manov [Z]).** *Any finite-dimensional simple Novikov algebra over an algebraically closed field  $\mathbb{F}$  with characteristic 0 isomorphic to  $(\mathbb{F}, \cdot)$ .*

Based on Osborn's work [O1], I obtained the following result:

**Theorem 2.2 (Xu, [X1]).** *A finite-dimensional simple Novikov algebra over an algebraically closed field  $\mathbb{F}$  with characteristic  $p > 2$  is either one-dimensional with a base element  $e$  such that  $e \circ e = e$  or has dimension  $p^k$  for some positive integer  $k$  with a basis  $\{\varsigma_j \mid j \in \overline{-1, p^k - 2}\}$  satisfying*

$$\varsigma_{j_1} \circ \varsigma_{j_2} = \binom{j_1+j_2+1}{j_2} \varsigma_{j_1+j_2} + \delta_{j_1,-1} \delta_{j_2,0} a \varsigma_{p^k-2} + \delta_{j_1,-1} \delta_{j_2,-1} b \varsigma_{p^k-2} \quad (2.2)$$

for  $j_1, j_2 \in \overline{-1, p^k - 1}$ , where  $a, b \in \mathbb{F}$  are constants.

Next, we consider infinite-dimensional simple Novikov algebra with  $\text{char } \mathbb{F} = 0$ . Let  $\Delta$  be an additive subgroup of  $\mathbb{F}$  and let  $\mathbb{F}_1$  be an extension field of  $\mathbb{F}$ . Take a map  $f : \Delta \times \Delta \rightarrow \mathbb{F}_1^\times = \mathbb{F}_1 \setminus \{0\}$  such that

$$f(\alpha, \beta) = f(\beta, \alpha), \quad f(\alpha, \beta)f(\alpha + \beta, \gamma) = f(\alpha, \beta + \gamma)f(\beta, \gamma) \quad (2.3)$$

for  $\alpha, \beta, \gamma \in \Delta$ . Take  $J$  to be the additive semi-group  $\{0\}$  or  $\mathbb{N}$ . Let  $\mathcal{A}(\Delta, f, J)$  be a vector space over  $\mathbb{F}_1$  with a basis

$$\{u_{\alpha,i} \mid \alpha \in \Delta, i \in \mathbb{N}\}. \quad (2.4)$$

We define an algebraic operation “ $\cdot$ ” on  $\mathcal{A}(\Delta, f, J)$  by

$$u_{\alpha,i} \cdot u_{\beta,j} = f(\alpha, \beta) u_{\alpha+\beta, i+j} \quad \text{for } \alpha, \beta \in \Delta, i, j \in J. \quad (2.5)$$

Then  $(\mathcal{A}(\Delta, f, J), \cdot)$  forms a commutative associative algebra over  $\mathbb{F}_1$ , and also over  $\mathbb{F}$ . We define a derivation  $\partial$  of  $\mathcal{A}(\Delta, f, J)$  over  $\mathbb{F}_1$  by

$$\partial(u_{\alpha,i}) = \alpha u_{\alpha,i} + i u_{\alpha, i-1} \quad \text{for } \alpha \in \Delta, i \in J, \quad (2.6)$$

where we treat

$$u_{\beta,j} = 0 \quad \text{if } (\beta, j) \notin \Delta \times J. \quad (2.7)$$

Note that  $\partial$  is also a derivation of  $\mathcal{A}(\Delta, f, J)$  over  $\mathbb{F}$ .

**Theorem 2.3 (Xu, [X2]).** *For any element  $\xi \in (\mathcal{A}(\Delta, f, J))$ , the Novikov algebra  $(\mathcal{A}(\Delta, f, J), \circ_\xi)$  (cf. (2.1)) is simple.*

Osborn [O3] gave a classification of infinite-dimensional simple Novikov algebras with an element  $e$  such that  $e \circ e \in \mathbb{F}e$ , assuming the existence of generalized-eigenspace decomposition

with respect to its left multiplication operator. There are four fundamantal mistakes in his classification. The first is using of Proposition 2.6 (d) in [O1] with  $\beta \neq 0$ , which was misproved. The second is that the eigenspace  $A_0$  in Lemma 2.12 of [O3] does not form a field when  $b = 0$  with respect to the Novikov algebraic operation. The third is that  $A_0$  may not be a perfect field when  $b \neq 0$ . The fourth is that the author forgot the case  $b = 0$  and  $\Delta = \{0\}$  in Lemma 2.8. In addition to these four mistakes, there are gaps in the arguments of classification in [O3]. It seems that one can not draw any conclusions of classification based on the arguments in [O3].

A linear transformation  $T$  of a vector space  $V$  is called *locally finite* if the subspace

$$\sum_{m=0}^{\infty} \mathbb{F}T^m(v) \text{ is finite-dimensional for any } v \in V. \quad (2.8)$$

An element  $u$  of a Novikov algebra  $\mathcal{N}$  is called *left locally finite* if its left multiplication operator  $L_u$  is locally finite.

I have re-established the following classification theorem:

**Theorem 2.4 (Xu, [X6])** *Let  $(\mathcal{N}, \circ)$  be an infinite-dimensional simple Novikov algebra over an algebraically field  $\mathbb{F}$  with characteristic 0. Suppose that  $\mathcal{N}$  contains a left locally finite element  $e$  whose right multiplication  $R_e$  is a constant map and left multiplication is surjective if  $R_e = 0$ . Then there exist an additive subgroup  $\Delta$  of  $\mathbb{F}$ , an extension field  $\mathbb{F}_1$  of  $\mathbb{F}$ , a map  $f : \Delta \times \Delta \rightarrow \mathbb{F}_1^\times$  satisfying (2.3) and  $\xi \in \mathbb{F}$  such that the algebra  $(\mathcal{N}, \circ)$  is isomorphic to  $(\mathcal{A}(\Delta, f, J), \circ_\xi)$ .*

Now we consider representations of a Novikov algebra. A module  $M$  of a Novikov algebras  $(\mathcal{N}, \circ)$  is a vector space with two linear maps

$$\mathcal{N} \times M \rightarrow M : (u, w) \mapsto u \circ w, \quad M \times \mathcal{N} \rightarrow M : (w, u) \mapsto w \circ u \quad (2.9)$$

such that (1.11) and (1.12) hold when one of the elements in  $\{u, v, w\}$  is in  $M$  and the other two are in  $\mathcal{N}$ . The commutator algebra associated with a finite-dimensional simple Novikov algebra over an algebraically closed field  $\mathbb{F}$  with characteristic  $p > 2$  is a rank-one simple Lie algebra of Witt type (cf. [O1]), whose finite-dimensional irreducible modules has not been completely classified yet. However, we have the following complete result.

**Theorem 2.5 (Xu, [X1]).** *Suppose that  $\mathbb{F}$  is an algebraically closed field with characteristic  $p > 2$ . If  $M$  is a finite-dimensional irreducible module of a finite-dimensional simple Novikov algebra  $(\mathcal{N}, \circ)$  in the presentation (2.2), then there exists a constant  $\lambda$  and a basis  $\{v_j \mid j \in \overline{-1, p^k - 2}\}$  of  $M$  such that*

$$\varsigma_{j_1} \circ v_{j_2} = \binom{j_1+j_2+1}{j_2} v_{j_1+j_2} + \binom{j_1+j_2+2}{j_1+1} \lambda v_{j_1+j_2+1} + \delta_{j_1, -1} \delta_{j_2, 0} a v_{p^k-2} \quad (2.10)$$

$$v_{j_2} \circ \varsigma_{j_1} = \binom{j_1+j_2+1}{j_1} v_{j_1+j_2} + \delta_{j_2,-1} \delta_{j_1,0} a v_{p^k-2} + \delta_{j_2,-1} \delta_{j_1,-1} b v_{p^k-2} \quad (2.11)$$

for  $j_1, j_2 \in \overline{-1, p^k - 2}$ . Moreover,  $\lambda \neq 0$  if  $a \neq 0$ . Conversely, (2.10) and (2.11) define an irreducible module for any  $0 \neq \lambda \in \mathbb{F}$  and for  $\lambda = 0$  if  $a = 0$ .

Let us go to the case of  $\text{char } \mathbb{F} = 0$ . First we shall give a construction of irreducible modules. Take  $J$  to be the additive semi-group  $\{0\}$  or  $\mathbb{N}$ . Let  $\mathcal{A}$  be a vector space with a basis

$$\{u_{\alpha,i} \mid \alpha \in \mathbb{F}, i \in J\}. \quad (2.12)$$

Define the operation “ $\cdot$ ” on  $\mathcal{A}$  by

$$u_{\alpha,i} \cdot u_{\beta,j} = u_{\alpha+\beta,i+j} \quad \text{for } \alpha, \beta \in \mathbb{F}, i, j \in J. \quad (2.13)$$

Then  $(\mathcal{A}, \cdot)$  forms a commutative associative algebra with the identity element  $1 = u_{0,0}$ . We define the derivation  $\partial$  on  $\mathcal{A}$  by (2.6). Let  $\Delta$  be an additive subgroup of  $\mathbb{F}$  such that  $J + \Delta \neq \{0\}$ . Set

$$\mathcal{N} = \sum_{\alpha \in \Delta, i \in J} \mathbb{F} u_{\alpha,i}. \quad (2.14)$$

For any fixed element  $\xi \in \mathcal{N}$ , we define the operation “ $\circ$ ” on  $\mathcal{A}$  by

$$u \circ v = u \cdot \partial(v) + \xi \cdot u \cdot v \quad \text{for } u, v \in \mathcal{A}. \quad (2.15)$$

By Theorem 2.3,  $(\mathcal{A}, \circ)$  forms a simple Novikov algebra and  $(\mathcal{N}, \circ)$  forms a simple subalgebra of  $(\mathcal{A}, \circ)$ . For  $\lambda \in \mathbb{F}$ , we set

$$M(\lambda) = \sum_{\alpha \in \Delta, i \in J} \mathbb{F} u_{\alpha+\lambda,i} \quad (2.16)$$

Expression (2.15) shows

$$\mathcal{N} \circ M(\lambda), M(\lambda) \circ \mathcal{N} \subset M(\lambda). \quad (2.17)$$

Thus  $M(\lambda)$  forms an  $\mathcal{N}$ -module. By the proof of Theorem 2.9 in [X2], we have

**Theorem 2.6 (Xu, [X6]).** *The  $\mathcal{N}$ -module  $M(\lambda)$  is irreducible.*

A natural question is to what extent the modules  $\{M(\lambda) \mid \lambda \in \mathbb{F}\}$  cover the irreducible modules of  $\mathcal{N}$ . Up to this point, we are not be able to answer this for a general element  $\xi \in \mathcal{N}$ . For an  $\mathcal{N}$ -module  $M$ , we define the left action  $L_M(u_{0,0})$  of  $u_{0,0}$  by

$$L_M(u_{0,0})(w) = u_{0,0} \circ w \quad \text{for } w \in M. \quad (2.18)$$

**Theorem 2.7 (Xu, [X6]).** *If  $\xi \in \mathbb{F}$ , then any irreducible  $\mathcal{N}$ -module  $M$  with locally finite  $L_M(u_{0,0})$  is isomorphic to  $M(\lambda)$  for some  $\lambda \in \mathbb{F}$ , when  $\mathbb{F}$  is algebraically closed.*

### 3 Constructions of Gel'fand-Dorfman Bialgebras

Recall the compatibility condition (1.17). Let  $(\mathcal{N}, \circ)$  be a Novikov algebra. Define

$$[u, v]^- = u \circ v - v \circ u \quad \text{for } u, v \in \mathcal{N}. \quad (3.1)$$

**Theorem 3.1 (Gel'fand and Dorfman, [GDo]).** *The triple  $(\mathcal{N}, [\cdot, \cdot]^-, \circ)$  forms a Gel'fand-Dorfman bialgebra.*

Let  $(\mathcal{A}, \cdot)$  be a commutative associative algebra. Denote by  $\text{Der } \mathcal{A}$  the Lie algebras of all the derivations of  $\mathcal{A}$ . Set

$$\mathcal{N} = \text{Der } \mathcal{A} \oplus \mathcal{A}. \quad (3.2)$$

We define two algebraic operation  $[\cdot, \cdot]$  and  $\circ$  on  $\mathcal{N}$  by

$$[d_1 + \xi_1, d_2 + \xi_2] = [d_1, d_2] + d_1(\xi_2) - d_2(\xi_1), \quad (3.3)$$

$$(d_1 + \xi_1) \circ (d_2 + \xi_2) = \xi_2(d_1 + \xi_1) \quad (3.4)$$

for  $d_1, d_2 \in \text{Der } \mathcal{A}$  and  $\xi_1, \xi_2 \in \mathcal{A}$ .

**Theorem 3.2 (Xu, [X4]).** *The triple  $(\mathcal{N}, [\cdot, \cdot], \circ)$  forms a Gel'fand-Dorfman bialgebra.*

The above construction is extracted from the simple Lie algebras of Witt type.

Our second construction is related to the following concept. A *Lie-Poisson algebra* is a vector space  $\mathcal{A}$  with two algebraic operations “ $\cdot$ ” and  $[\cdot, \cdot]$  such that  $(\mathcal{A}, \cdot)$  forms a commutative associative algebra,  $(\mathcal{A}, [\cdot, \cdot])$  forms a Lie algebra and the following compatibility condition is satisfied:

$$[u, v \cdot w] = [u, v] \cdot w + v \cdot [u, w] \quad \text{for } u, v, w \in \mathcal{A}. \quad (3.5)$$

Let  $(\mathcal{A}, \cdot, [\cdot, \cdot])$  be a Lie-Poisson algebra and let  $\partial$  be a derivation of the algebra  $(\mathcal{A}, \cdot)$  such that

$$\partial[u, v] = [\partial(u), v] + [u, \partial(v)] + \xi[u, v] \quad \text{for } u, v \in \mathcal{A}, \quad (3.6)$$

where  $\xi \in \mathbb{F}$  is a constant. Now we define another algebraic operation  $\circ$  on  $\mathcal{A}$  by

$$u \circ v = u\partial(v) + \xi uv \quad \text{for } u \in \mathcal{A}, v \in \mathcal{A}. \quad (3.7)$$

**Theorem 3.3 (Xu, [X4]).** *The triple  $(\mathcal{A}, [\cdot, \cdot], \circ)$  forms a Gel'fand-Dorfman bialgebra.*

The above construction is related to the Lie algebras of Hamiltonian type and Contact type.

Let  $(\mathcal{A}, \cdot)$  be a commutative associative algebra and let  $\partial_1, \partial_2$  be mutually commutative derivations of  $(\mathcal{A}, \cdot)$ . Define algebraic operations on  $\mathcal{A}$  by

$$[u, v] = \partial_1(u)\partial_2(v) - \partial_2(u)\partial_1(v) + u\partial_2(v) - \partial_2(u)v, \quad u \circ v = u\partial_2(v) \quad (3.8)$$

for  $u, v \in \mathcal{A}$ .

**Theorem 3.4 (Xu, [X4]).** *The triple  $(\mathcal{A}, [\cdot, \cdot], \circ)$  forms a Gel'fand-Dorfman bialgebra.*

Let  $(\mathcal{A}, \cdot)$ ,  $\partial_1$  and  $\partial_2$  be the same as in the above. For any constant  $b \in \mathbb{F}$ , we define

$$[u, v] = \partial_1(u)\partial_2(v) - \partial_2(u)\partial_1(v) + b(u\partial_2(v) - \partial_2(u)v), \quad u \circ v = u\partial_1(v) + buv \quad (3.9)$$

for  $u, v \in \mathcal{A}$ .

**Theorem 3.5 (Xu, [X4]).** *The triple  $(\mathcal{A}, [\cdot, \cdot], \circ)$  forms a Gel'fand-Dorfman bialgebra.*

## 4 Classifications of the Bialgebras

If  $(\mathcal{N}, [\cdot, \cdot], \circ)$  is a Gel'fand-Dorfman bialgebra, then we say that  $(\mathcal{N}, [\cdot, \cdot])$  is a *Lie algebra over the Novikov algebra*  $(\mathcal{N}, \circ)$ . Similarly,  $(\mathcal{N}, \circ)$  is a *Novikov algebra over the Lie algebra*  $(\mathcal{N}, [\cdot, \cdot])$ .

Let  $\mathbb{F}$  be an algebraically closed field with  $\text{char } \mathbb{F} = 0$ . Take  $J$  to be the additive semi-group  $\{0\}$  or  $\mathbb{N}$  and take an additive subgroup  $\Delta$  of  $\mathbb{F}$ . Let  $\mathcal{A}(\Delta, J)$  be a vector space with a basis

$$\{u_{\alpha, i} \mid \alpha \in \Delta, i \in J\}. \quad (4.1)$$

For any constant  $b \in \mathbb{F}$ , we define an algebraic operation  $\circ_b$  on  $\mathcal{A}(\Delta, J)$  by

$$u \circ_b v = u \cdot \partial(v) + b \cdot u \cdot v \quad \text{for } u, v \in \mathcal{A}(\Delta, J). \quad (4.2)$$

By Theorem 2.3,  $(\mathcal{A}(\Delta, J), \circ_b)$  forms a simple Novikov algebra.

**Theorem 4.1 (Osborn and Zel'manov, [OZ]).** *If  $\Delta = \mathbb{Z}$ ,  $J = \{0\}$ ,  $b \notin \mathbb{Z}$  or  $\Delta = 0$ ,  $J = \mathbb{N}$ , then any nontrivial Lie algebra over the Novikov algebra  $(\mathcal{A}(\Delta, J), \circ_b)$  is isomorphic to  $(\mathcal{A}(\Delta, J), [\cdot, \cdot]^-)$ , where*

$$[u_{\alpha, i}, u_{\beta, j}]^- = (\beta - \alpha)u_{\alpha+\beta, i+j} + (j - i)u_{\alpha+\beta, i+j-1} \quad (4.3)$$

for  $\alpha, \beta \in \Delta$  and  $i, j \in J$ .

**Theorem 4.2 (Xu, [X4]).** *If  $b \notin \Delta$ , then any Lie algebra over the Novikov algebra  $(\mathcal{A}(\Delta, J), \circ_b)$  has the Lie bracket*

$$\begin{aligned} [u_{\alpha, i}, u_{\beta, j}] &= [(\alpha + b)\varphi(\beta) - (\beta + b)\varphi(\alpha)]u_{\alpha+\beta, i+j} \\ &\quad + [i(\varphi(\beta) - \lambda(\beta + b)) + j(\lambda(\alpha + b) - \varphi(\alpha))]u_{\alpha+\beta, i+j-1} \end{aligned} \quad (4.4)$$

for  $\alpha, \beta \in \Delta$ ,  $i, j \in J$ , where  $\varphi : \Delta \rightarrow \mathbb{F}$  is an additive group homomorphism and  $\lambda \in \mathbb{F}$  is a constant.



When  $J = \{0\}$ , (4.4) gives a Block algebra (cf. [B]). Next we assume  $J = \{0\}$  and redenote  $u_{\alpha,0}$  by  $u_\alpha$  for  $\alpha \in \Delta$ .

**Theorem 4.3 (Xu, [X4])** *Any Lie algebra over the Novikov algebra  $(\mathcal{A}(\Delta, \{0\}), \circ_0)$  has the Lie bracket*

$$[u_\alpha, u_\beta] = (\phi(\alpha, \beta) + a(\beta - \alpha))u_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta, \quad (4.5)$$

where  $a$  is a constant and  $\phi(\cdot, \cdot) : \Delta \times \Delta \rightarrow \mathbb{F}$  is a skew-symmetric map such that there exists a symmetric map  $S_0(\cdot, \cdot, \cdot) : \Delta \times \Delta \times \Delta \rightarrow \mathbb{F}$  satisfying

$$\phi(\beta + \gamma, \alpha) = \phi(\gamma, \alpha) + \phi(\beta, \alpha) + \alpha S_0(\alpha, \beta, \gamma), \quad (4.6)$$

$$(\gamma\phi(\alpha, \beta) + \alpha\phi(\beta, \gamma) + \beta\phi(\gamma, \alpha))(S_0(\alpha, \beta, \gamma) - a) = 0. \quad (4.7)$$

for  $\alpha, \beta, \gamma \in \Delta$ . In particular, the following  $\phi$  satisfies our condition: when  $a = 0$ ,  $\phi$  is any skew-symmetric  $\mathbb{Z}$ -bilinear form; when  $a \neq 0$ ,

$$\phi(\alpha, \beta) = \alpha\varphi_0(\beta) - \beta\varphi_0(\alpha) \quad \text{for } \alpha, \beta \in \Delta, \quad (4.8)$$

where  $\varphi_0 : \Delta \rightarrow \mathbb{F}$  is an additive group homomorphism.

**Theorem 4.4 (Xu, [X4])** *Suppose  $0 \neq b \in \Delta$ . Any Lie algebra over the Novikov algebra  $(\mathcal{A}(\Delta, \{0\}), \circ_b)$  has the Lie bracket*

$$[u_\alpha, u_\beta] = \theta(\alpha, \beta)u_{\alpha+\beta+b} + ((\alpha + b)\varphi(\beta) - (\beta + b)\varphi(\alpha))u_{\alpha+\beta} \quad (4.9)$$

for  $\alpha, \beta \in \Delta$ , where  $\varphi : \Delta \rightarrow \mathbb{F}$  is additive group homomorphism,  $\theta \equiv 0$  if  $\varphi(b) \neq 0$  and  $\theta : \Delta \times \Delta \rightarrow \mathbb{F}$  is a skew-symmetric map satisfying

$$(\alpha + b)(\theta(\alpha + \gamma, \beta) - \theta(\gamma, \beta) - \theta(\alpha, \beta)) = (\beta + b)(\theta(\beta + \gamma, \alpha) - \theta(\gamma, \alpha) - \theta(\beta, \alpha)) \quad (4.10)$$

and

$$\theta(\alpha, \beta)\theta(\alpha + \beta + b, \gamma) + \theta(\beta, \gamma)\theta(\beta + \gamma + b, \alpha) + \theta(\gamma, \alpha)\theta(\gamma + \alpha + b, \beta) = 0 \quad (4.11)$$

if  $\varphi(b) = 0$ . In particular, we can take  $\varphi : \Delta \rightarrow \mathbb{F}$  to be any additive group homomorphism such that  $\varphi(b) = 0$  and take  $\theta(\cdot, \cdot)$  to be a skew-symmetric  $\mathbb{Z}$ -bilinear map such that  $b \in \text{Rad}_\theta$  or

$$\theta(\alpha, \beta) = \varphi_1(\alpha)\varphi_2(\beta) - \varphi_1(\beta)\varphi_2(\alpha) \quad \text{for } \alpha, \beta \in \Delta, \quad (4.12)$$

where  $\varphi_1, \varphi_2 : \Delta \rightarrow \mathbb{F}$  are additive group homomorphisms such that  $\varphi_1(b) = 0$  and  $\varphi_2(b) \neq 0$ .

**Theorem 4.5 (Xu, [X4]).** *If  $\circ$  is a Novikov algebra operation on the space  $\mathcal{A}(\Delta, \{0\})$  such that*

$$u_\alpha \circ u_\beta - u_\beta \circ u_\alpha = (\beta - \alpha)u_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta, \quad (4.13)$$

then there exists an element  $\xi \in \mathcal{A}(\Delta, \{0\})$  such that

$$u_\alpha \circ u_\beta = (\beta + \xi)u_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \Delta. \quad (4.14)$$

## 5 Application to Cubic and Quartic Conformal Algebras

For two vector spaces  $V_1$  and  $V_2$ , we denote by  $LM(V_1, V_2)$  the space of linear maps from  $V_1$  to  $V_2$ . We shall also use the following operator of taking residue:

$$\text{Res}_z(z^n) = \delta_{n,-1} \quad \text{for } n \in \mathbb{Z}. \quad (5.1)$$

Furthermore, all the binomials are assumed to be expanded in the nonnegative powers of the second variable. For example ,

$$\frac{1}{z-x} = \frac{1}{z(1-x/z)} = \sum_{j=0}^{\infty} z^{-1} \left(\frac{x}{z}\right)^j = \sum_{j=0}^{\infty} z^{-j-1} x^j. \quad (5.2)$$

In particular, the above equation implies

$$\text{Res}_x \frac{1}{z-x} \left( \sum_{j \in \mathbb{Z}} \xi_j z^j \right) = \sum_{j=1}^{\infty} \xi_{-j} z^{-j}. \quad (5.3)$$

So the operator  $\text{Res}_x(1/(z-x))(\text{---})$  is taking the part of negative powers in a formal series and changing the variable  $x$  to  $z$ .

A conformal algebra  $R$  is an  $\mathbb{F}[\partial]$ -module equipped with a linear map  $Y^+(\cdot, z) : R \rightarrow LM(R, R[z^{-1}]z^{-1})$  satisfying:

$$Y^+(\partial u, z) = \frac{d}{dz} Y^+(u, z), \quad Y^+(u, z)v = \text{Res}_x \frac{e^{x\partial} Y^+(v, -x)u}{z-x}, \quad (5.4)$$

$$Y^+(u, z_1)Y^+(v, z_2) - Y^+(v, z_2)Y^+(u, z_1) = \text{Res}_x \frac{Y^+(Y^+(u, z_1-x)v, x)}{z_2-x} \quad (5.5)$$

for  $u, v \in R$ . We denote by  $(R, \partial, Y^+(\cdot, z))$  a conformal algebra.

The above definition is the equivalent generating-function form to that given in [K], where the author used the component formula with  $Y^+(u, z) = \sum_{n=0}^{\infty} u_{(n)} z^{-n-1}/n!$ . The connection between the Lie algebra with one-variable structure in (1.3) and conformal algebra is that  $R = \mathbb{F}[\partial] \otimes_{\mathbb{F}} V$  and

$$Y^+(u, z)v = \sum_{i=0}^m \sum_{j=0}^n (-1)^i i! \partial^j w_{ij} z^{-i-1}. \quad (5.6)$$

Conformal algebras are equivalent to linear Hamiltonian operators (cf. [X5]).

Suppose that  $(R, \partial, Y^+(\cdot, z))$  is a conformal algebra that is a free  $\mathbb{F}[\partial]$ -module over a subspace  $V$ , namely

$$R = \mathbb{F}[\partial]V \quad (\cong \mathbb{F}[\partial] \otimes_{\mathbb{F}} V). \quad (5.7)$$

Let  $m$  be a positive integer. The algebra  $R$  is called *of degree  $m$*  if for any  $u, v \in V$ ,

$$Y^+(u, z)v = \sum_{0 < j; i+j \leq m} \partial^i w_{i,j} z^{-j} \quad \text{with } w_{i,j} \in V, \quad (5.8)$$

and  $w_{m-j,j} \neq 0$  for some  $u, v \in V$  and  $j \in \overline{1, m}$ . A *quadratic conformal algebra* is a conformal algebra of degree 2, a *cubic conformal algebra* is a conformal algebra of degree 3 and a *quartic conformal algebra* is a conformal algebra of degree 4. In [X4], I have proved that a quadratic conformal algebra is equivalent to a Gel'fand-Dorfman bialgebra. Below I will give the construction of quadratic conformal algebra from Gel'fand-Dorfman bialgebras.

Let  $(\mathcal{N}, [\cdot, \cdot], \circ)$  be a Gel'fand Dorfman bialgebra. Set

$$R_{\mathcal{N}} = \mathbb{F}[\partial] \otimes_{\mathbb{F}} \mathcal{N}. \quad (5.9)$$

So  $R_{\mathcal{N}}$  is a free  $\mathbb{F}[\partial]$ -module generated by  $\mathcal{N}$ . For convenience, we identify  $\mathcal{N}$  with  $1 \otimes \mathcal{N}$ . We define a linear map  $Y^+(\cdot, z) : R \rightarrow LM(R, R[z^{-1}]z^{-1})$  by

$$Y^+(\partial^m u, z)\partial^n v = \sum_{j=0}^n (-1)^j \binom{n}{j} \left( \frac{d}{dz} \right)^{m+j} \partial^{n-j} ([v, u] + \partial(v \circ u)) z^{-1} + (u \circ v + v \circ u) z^{-2} \quad (5.10)$$

for  $u, v \in \mathcal{N}$  and  $m, n \in \mathbb{N}$ .

**Theorem 5.1 (Xu, [X4]).** *The triple  $(R_{\mathcal{N}}, \partial, Y^+(\cdot, z))$  forms a quadratic conformal algebra.*

Next I will use the above theorem to construct simple cubic conformal algebras and quartic conformal algebras. A conformal algebra  $(R, \partial, Y^+(\cdot, z))$  is called *simple* if there does not exist a nonzero proper subspace  $\mathcal{I}$  of  $R$  such that

$$Y^+(u, z)(\mathcal{I}) \subset \mathcal{I}[z^{-1}] \quad \text{for } u \in R. \quad (5.11)$$

Take  $J$  to be the additive semi-group  $\{0\}$  or  $\mathbb{N}$ . Let  $\Gamma$  be an additive subgroup of  $\mathbb{F}^2$  such that

$$(J, 0) + \Gamma \not\subset (0, \mathbb{F}), \quad \Gamma \not\subset (\mathbb{F}, 0). \quad (5.12)$$

Let  $\mathcal{A}$  be the semi-group algebra of  $\Gamma \times J$  with the canonical basis  $\{u_{\alpha, i} \mid \alpha \in \Delta, i \in J\}$ , that is,

$$u_{\alpha, i} \cdot u_{\beta, j} = u_{\alpha+\beta, i+j} \quad \text{for } \alpha, \beta \in \Gamma, i, j \in J. \quad (5.13)$$

Define two derivations  $\partial_1$  and  $\partial_2$  of  $\mathcal{A}$  by

$$\partial_1(u_{\alpha,i}) = \alpha_1 u_{\alpha,i} + i u_{\alpha,i-1}, \quad \partial_2(u_{\alpha,i}) = \alpha_2 u_{\alpha,i} \quad (5.14)$$

for  $\alpha = (\alpha_1, \alpha_2) \in \Gamma$  and  $i \in J$ .

First we define

$$[u, v]_1 = \partial_1(u)\partial_2(v) - \partial_2(u)\partial_1(v) + u\partial_2(v) - \partial_2(u)v, \quad u \circ_1 v = u\partial_2(v) \quad (5.15)$$

Then  $(\mathcal{A}, [\cdot, \cdot]_1, \circ_1)$  forms a Gel'fand-Dorfman bialgebra by Theorem 3.4. On  $R_{\mathcal{A}}$ , the structure map (5.10) is determined by

$$\begin{aligned} Y_1^+(u_{\alpha,i}, z)u_{\beta,j} &= [((1 + \alpha_1)\beta_2 - \alpha_2(1 + \beta_1))u_{\alpha+\beta,i+j} + (i\beta_2 - j\alpha_2)u_{\alpha+\beta,i+j-1} \\ &\quad + \beta_2\partial u_{\alpha+\beta,i+j}]z^{-1} + (\alpha_2 + \beta_2)u_{\alpha+\beta,i+j}z^{-2}. \end{aligned} \quad (5.16)$$

for  $\alpha = (\alpha_1, \alpha_2)$ ,  $\beta = (\beta_1, \beta_2) \in \Gamma$  and  $i, j \in J$ . When  $\beta_2 = -\alpha_2$ , (5.16) becomes

$$Y_1^+(u_{\alpha,i}, z)u_{\beta,j} = \beta_2[(\alpha_1 + \beta_1 + 2 + \partial)u_{\alpha+\beta,i+j} + (i + j)u_{\alpha+\beta,i+j-1}]z^{-1}. \quad (5.17)$$

Set

$$R_1 = \sum_{\alpha=(\alpha_1, \alpha_2) \in \Gamma, \alpha_2 \neq 0} \mathbb{F}[\partial]u_{\alpha,i} + \sum_{\beta=(\beta_1, 0) \in \Gamma, j \in J} \mathbb{F}[\partial][(\beta_1 + 2 + \partial)u_{\beta,j} + ju_{\beta,j-1}]. \quad (5.18)$$

Then  $R_1$  forms a subalgebra of  $(R_{\mathcal{A}}, \partial, Y_1^+(\cdot, z))$  by (5.17), that is,

$$\partial R_1 \subset R_1, \quad Y_1^+(u, z)v \in R_1[z^{-1}] \quad \text{for } u, v \in R_1. \quad (5.19)$$

**Theorem 5.2 (Xu, [X7]).** *The conformal algebra  $(R_1, \partial, Y_1^+(u, z))$  is a simple cubic conformal algebra.*

Let  $\mathcal{A}$ ,  $\partial_1$  and  $\partial_2$  be the same as in the above. Take  $b \in \mathbb{F}$  to be any fixed constant such that

$$(\mathbb{F}, 2b) \bigcap \Gamma \neq \emptyset, \quad \Gamma \not\subset (\mathbb{F}, 0). \quad (5.20)$$

We define

$$[u, v]_2 = \partial_2(u)\partial_1(v) - \partial_1(u)\partial_2(v) + b(u\partial_1(v) - \partial_1(u)v), \quad u \circ_2 v = u\partial_2(v) + buv \quad (5.21)$$

for  $u, v \in \mathcal{A}$ .

Then  $(\mathcal{A}, [\cdot, \cdot]_2, \circ_2)$  forms a Gel'fand-Dorfman bialgebra by Theorem 3.5. On  $R_{\mathcal{A}}$ , the structure map (5.10) is determined by

$$\begin{aligned} Y_2^+(u_{\alpha,i}, z)u_{\beta,j} &= [(\alpha_1(\beta_2 + b) - (\alpha_2 + b)\beta_1)u_{\alpha+\beta,i+j} + (i(\beta_2 + b) - j(\alpha_2 + b))u_{\alpha+\beta,i+j-1} \\ &\quad + (\beta_2 + b)\partial u_{\alpha+\beta,i+j}]z^{-1} + (\alpha_2 + \beta_2 + 2b)u_{\alpha+\beta,i+j}z^{-2}. \end{aligned} \quad (5.22)$$

for  $\alpha = (\alpha_1, \alpha_2)$ ,  $\beta = (\beta_1, \beta_2) \in \Gamma$  and  $i, j \in J$ . When  $\alpha_2 = 2b - \beta_2$ , (5.22) becomes

$$Y_2^+(u_{\alpha,i}, z)u_{\beta,j} = (\beta_2 + b)[(\alpha_1 + \beta_1 + \partial)u_{\alpha+\beta, i+j} + (i+j)u_{\alpha+\beta, i+j-1}]z^{-1}. \quad (5.23)$$

Set

$$R_2 = \sum_{\alpha=(\alpha_1, \alpha_2) \in \Gamma, \alpha_2 \neq -2b} \mathbb{F}[\partial]u_{\alpha,i} + \sum_{\beta=(\beta_1, -2b) \in \Gamma, j \in J} \mathbb{F}[\partial][(\beta_1 + \partial)u_{\beta,j} + ju_{\beta,j-1}]. \quad (5.24)$$

Then  $R_2$  forms a subalgebra of  $(R_{\mathcal{A}}, \partial, Y_2^+(\cdot, z))$  by (5.23), that is,

$$\partial R_2 \subset R_2, \quad Y_2^+(u, z)v \in R_2[z^{-1}] \quad \text{for } u, v \in R_2. \quad (5.25)$$

**Theorem 5.3 (Xu, [X7]).** *The conformal algebra  $(R_2, \partial, Y_2^+(u, z))$  is a simple cubic conformal algebra if  $b = 0$  and is a simple quartic conformal algebra if  $b \neq 0$ .*

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